Collocation Method With GeoGebra in Linear Second Order Differential Equations

Jorge Olivares Funes, [1, a)] María Rojas -Medar, [1, b)] *and* Pablo Martin [2, c)]

[1]*Department of Mathematics, University of Antofagasta, Antofagasta, Chile*
[2]*Department of Physics, University of Antofagasta, Antofagasta, Chile*

[a)] *Correspondig autor:jorge.olivares@uantof.cl*
[b)] *maria.rojas@uantof.cl*
[c)] *pablo.martin@uantof.cl*

Abstract. In this article we will show how to find approximate solutions, using the numerical analysis differential placement method to the second order linear equations of the form $\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = Q(x), y(0) = y(a) = 0$, with the GeoGebra software, where "a" is a positive number. The use of GeoGebra in the numerical analysis allows us to view the solutions and approximations of the differential equations, simultaneously, interactively, and dynamically.

INTRODUCTION

Sometimes it is not so easy to solve differential equations of the form

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = Q(x), y(0) = y(a) = 0,$$

where "a" is a non-zero positive number. The values that the expressions $A(x), B(x)$ and $Q(x)$ take can make trying to find the solution more difficult than previously thought.

That is why we focus our attention on the free software GeoGebra, since it has tools to relate the graphical and algebraic aspects of the same mathematical object, see figures (1) and (2).

In this paper, we will start using the method of placing order 2, given by $U_2 = x(ax)(\ \alpha_1 + \alpha_2 x)$ to approximate the solution to the differential equation

$$\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = Q(x), \qquad y(0) = y(a) = 0$$

where $A\ and\ B$ are real constants, and numbers $\alpha_1\ y\ \alpha_2$ are numbers to be determined.

1. NUMERICAL DEVELOPMENT WITH GEOGEBRA

To begin, we will see, through examples of GeoGebra, applications such as the absolute error of the formal solution versus the approximation given by $U_2 = x(ax)(\ \alpha_1 + \alpha_2 x)$, for some values of "$a$".

Example 1

For:

$$\frac{d^2g}{dx^2} + g = -x, \qquad 0 \le x \le a$$

$$g(0) = g(a) = 0$$

The solution is $g = a\, csc(a) \sin(x) - x$

Considering $U_2 = x(ax)(\ \alpha_1 + \alpha_2 x)$, as the approximation polynomial, the error is given by:

$$\varepsilon = \frac{d^2U_2}{dx^2} + U_2 + x$$

⇨When replacing:

$\frac{\boldsymbol{d^2U_2}}{\boldsymbol{dx^2}} = -2\alpha_1 + 2\alpha_2 a - 6\ \alpha_2 xy\ U_2 = x(ax)(\ \alpha_1 + \alpha_2 x)$ in $\boldsymbol{\varepsilon}$ and simplifying.

If you have:

$$\alpha_1(-2 + ax - x^2) + \ \alpha_2(2a - 6x + ax^2 - x^3) + x = 0$$

with $x = \frac{a}{4}$

$$\alpha_1\left(-2 + \frac{3a^2}{16}\right) + \alpha_2\left(\frac{a}{2} + \frac{3a^3}{64}\right) + \frac{a}{4} = 0$$

and $x = \frac{a}{2}$

$\alpha_1\left(-2 + \frac{a^2}{4}\right) + \alpha_2\left(-a + \frac{a^3}{8}\right) + \ \frac{a}{2} = 0$

So we form the following system:

$$\alpha_1\left(-2+\frac{3a^2}{16}\right)+\alpha_2\left(\frac{a}{2}+\frac{3a^3}{64}\right)+\frac{a}{4}=0$$

$$\alpha_1\left(-2+\frac{a^2}{4}\right)+\alpha_2\left(-a+\frac{a^3}{8}\right)+\frac{a}{2}=0$$

whose solution is $\alpha_1=\frac{-2a^3+128a}{3a^4-120a^2+768}$

$$\alpha_2=\frac{-8a^2+128}{3a^4-120a^2+768}$$

Therefore $U_2=x(ax)\left(\frac{-2a^3+128a}{3a^4-120a^2+768}+\frac{-8a^2+128}{3a^4-120a^2+768}\,x\right)$

The parameter "a" assumes values between 0.1 and 3. Thus, for $a=0.7$, the highest absolute error is less than 0.05, and the error varies depending on the values assumed by "a" (figure 1). For $a\neq n\pi,\ n\epsilon\mathbb{Z}$

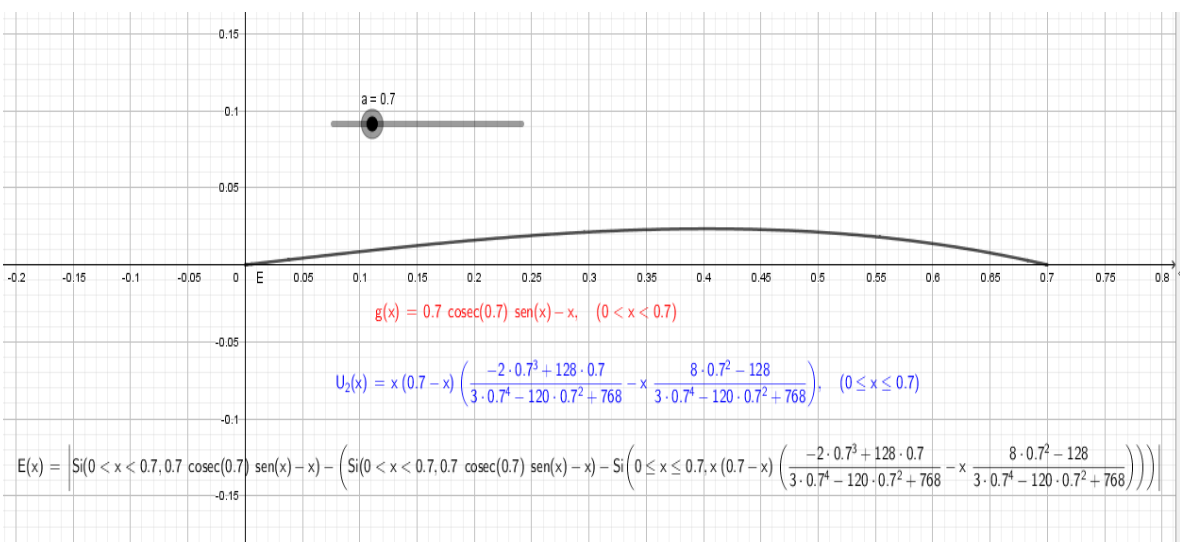

FIGURE 1.

Example 2

For:

$$\frac{d^2g}{dx^2}-g=a,\qquad 0\leq x\leq a$$

$$g(0) = g(a) = 0$$

The parameter "a" assumes values between 0.1 and 5.

The solution is

$$g = ax + \frac{a^2}{e^a - 1}e^x - \frac{a^2}{e^a - 1}$$

Considering $U_2 = x(ax)(\ \alpha_1 + \alpha_2 x)$ as the approximation polynomial, the error is given by

$$\varepsilon = \frac{d^2U_2}{dx^2} - \frac{dU_2}{dx} - a$$

⇒ When replacing: $\frac{d^2U_2}{dx^2} = -2\alpha_1 + 2\alpha_2 a - 6\alpha_2 xy \frac{dU_2}{dx} = a\alpha_1 + 2\alpha_2 ax - 2x\alpha_1 - 3\alpha_2 x^2$

in ε , and simplifying.

If you have:

$$-2\alpha_1 + \alpha_2(2a - 6x - 2ax + 3x^2) - a = 0$$

with $x = \frac{a}{4}$

$$\alpha_1\left(-2 - \frac{a}{2}\right) + \alpha_2\left(\frac{a}{2} - \frac{5a^3}{16}\right) - a = 0$$

and $x = \frac{a}{2}$

$$-2\alpha_1 + \alpha_2\left(-a - \frac{a^2}{4}\right) - a = 0$$

So we form the system

$$\alpha_1\left(-2-\frac{a}{2}\right)+\alpha_2\left(\frac{a}{2}-\frac{5a^3}{16}\right)-a=0$$

$$-2\alpha_1+\alpha_2\left(-a-\frac{a^2}{4}\right)-a=0$$

whose solution is $\alpha_1=\frac{a^2-24a}{2a^2+6a+48}$

$$\alpha_2=\frac{-4a}{a^2+3a+24}$$

Therefore $U_2=x(ax)\left(\frac{a^2-24a}{2a^2+6a+48}+\frac{-4a}{a^2+3a+24}\ x\right)$

Thus, for $a=0.5$, the highest absolute error is less than 0.02, and the error varies depending on the values assumed by "a" (figure 2).

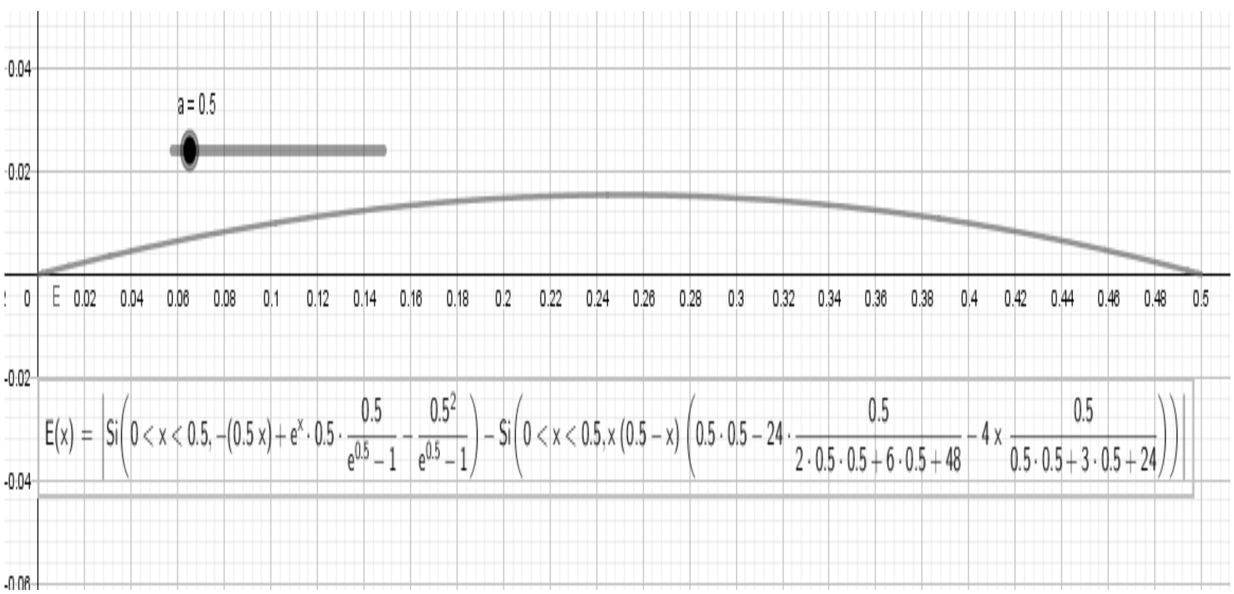

FIGURE 2.

Example 3

For:

$$\frac{d^2g}{dx^2}+g=1-\frac{x}{a},\qquad 0\le x\le a$$

$$g(0) = g(a) = 0$$

The parameter "a" assumes values between 0.1 and 2.

The solution is:

$$g = \frac{-x}{a} + cot\,(a)\sin(x) - \cos(x) + 1$$

Considering $U_2 = x(ax)(\,\alpha_1 + \alpha_2 x)$ the approximation polynomial, the error is given by

$$\varepsilon = \frac{d^2U_2}{dx^2} + U_2 - 1 + \frac{x}{a}$$

⇨ When replacing: $\frac{d^2U_2}{dx^2} = -2\alpha_1 + 2\alpha_2 a - 6\alpha_2 x$ y $U_2 = x(ax)(\,\alpha_1 + \alpha_2 x)$ in ε, and simplifying.

If you have:

$$\alpha_1(-2 + ax - x^2) + \alpha_2(2a - 6x + ax^2 - x^3) - 1 + \frac{x}{a} = 0$$

with $x = \frac{a}{4}$

$$\alpha_1\left(-2 + \frac{3a^2}{16}\right) + \alpha_2\left(\frac{a}{2} + \frac{3a^3}{64}\right) - \frac{3}{4} = 0$$

and $x = \frac{a}{2}$

$$\alpha_1\left(-2 + \frac{a^2}{4}\right) + \alpha_2\left(-a + \frac{a^3}{8}\right) - \frac{1}{2} = 0$$

whose solution is $\alpha_1 = \frac{186a^2 - 256}{45a^4 - 568a^2 + 768}$

$$\alpha_2 = \frac{-24a^2 + 128}{45a^5 - 568a^3 + 768a}$$

Therefore $U_2 = x(ax)\left(\frac{186a^2-256}{45a^4-568a^2+768} + \frac{-24a^2+128}{45a^5-568a^3+768a}\ x\right)$

The parameter “a” assumes values between 0.1 and 3.

Thus, for $a = 1.5$, the highest absolute error is less than or equal to 0.4 and the error varies depending on the values assumed by "a" (figure 3). For $a \neq n\pi, n \in \mathbb{Z}$

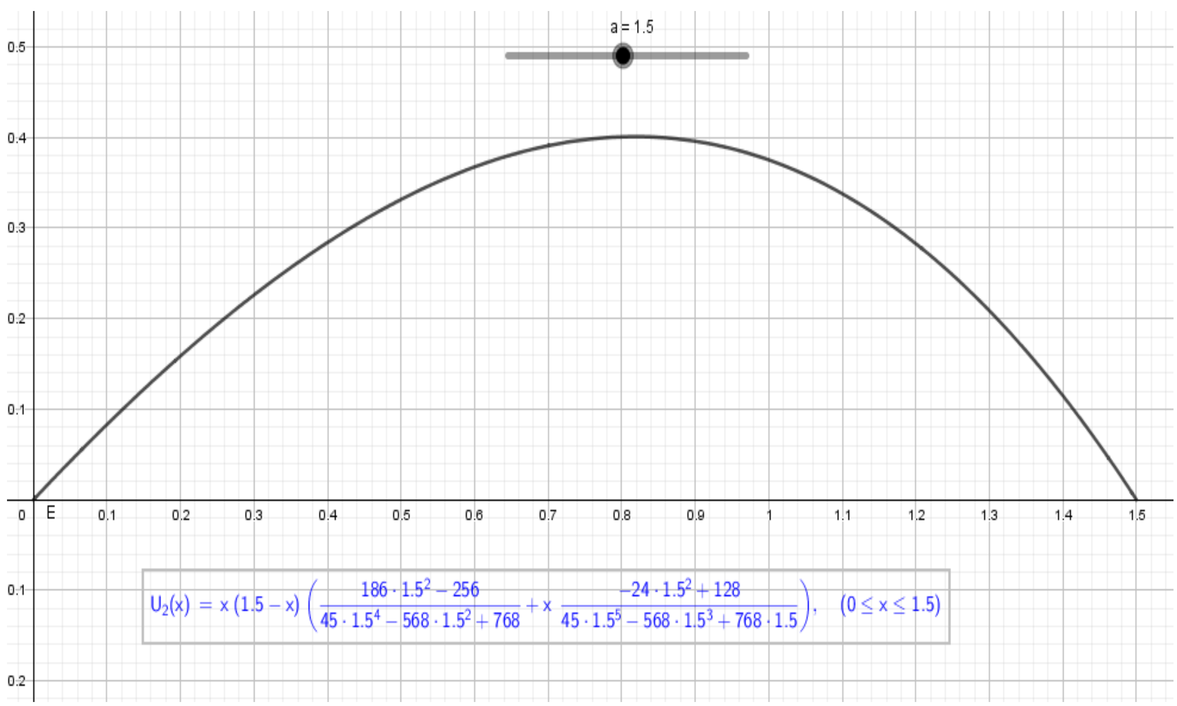

FIGURE 3.

CONCLUSION

We see that the use of GeoGebra in the numerical visualization of the absolute error in differential equations, is of great help, and motivator to expand this same idea to other methods of numerical analysis. The dynamism of the GeoGebra with the students, makes the learning of the Collocation Method a place to escape the traditional and usual way.

ACKNOWLEDGMENTS

This work was funded by Research Management Directorate at the University of Antofagasta -Chile. We thank the University of Antofagasta, especially the Teaching Innovation Management Area via the University Teaching Investigation Support Program (PAIDU) to which the University Teaching Investigation Core belongs (NIDU 2-001-22) called "Geogebra Applets for teaching Mathematics", whose members are responsible for the investigation which led to the present publication. Thank you for your support, accompaniment and formative program intended to promote investigation in Teaching and elevating the academic indicators of this institution.

REFERENCES

1. M. Graciela Caligaris, M. E. Schivo and M. R. Romiti, *Calculus & GeoGebra, an Interesting Partnership (*Procedia - Social and Behavioral Sciences , Volume 174, 12 February 2015, Pages 1183-1188). https://www.sciencedirect.com/science/article/pii/S1877042815007879
2. J. Olivares Funes and E. R. Valero Kari, *Finite Elements and Finite Differences in some differential equations of second linear order with GeoGebra (*Journal of Physics: Conference Series, Volume 1730, 2020). https://iopscience.iop.org/article/10.1088/1742-6596/1730/1/012068
3. J. Olivares Funes, P. Martin and E. Valero Kari, *Galarkin Method with GeoGebra in Differential Equations* (Journal of Physics: Conference Series, Volume 2090, 2021). https://iopscience.iop.org/article/10.1088/1742-6596/2090/1/012092
4. J. Olivares Funes and E. Valero Kari, *Visualizing some numerical solutions of linear second order differential equations with the Euler method and Lagrange interpolation via GeoGebra* (Journal of Physics: Conference Series, Volume 2090, 2021). https://iopscience.iop.org/article/10.1088/1742-6596/2090/1/012090